Deligne-Lusztig varieties

These are notes for a talk given by me (Vidhu Adhihetty) for a graduate student seminar on Deligne-Lusztig theory. Any mistakes are my own.

Introduction

We begin by setting some notation:

G^F is our finite group of Lie type, being the fixed points under Frobenius of some (connected) linear algebraic group G.

Let B be a fixed Borel subgroup of G, with a maximal torus T. These pairs are mutually conjugate, meaning if there is some other Borel B' which cotnains a maximal torus T', then there exists $g \in G$ such that $gBg^{-1} = B'$ and $gTg^{-1} = T'$.

Denote by $W := N_G(T)/T$ the Weyl group of your torus $T \subset B$. Note that all Weyl groups are isomorphic via conjugation, so we speak of *the* Weyl group of G.

We denote by X the flag variety of all Borel subgroups, and note that since they are mutually conjugate (and furthermore, $N_G(B) = B$), $X = G/B$. Recall from Amal's lecture that X is not a group scheme as B is not normal (unless X has only one element), but it is a projective variety.

Note that by Lang's theorem, since G is connected and it acts on X transitively, and furthermore, $F(B')$ is a Borel subgroup for any Borel B' (here F is the associated Frobenius), by Lang's theorem, we find that there is a Borel subgroup B' such that $F(B') = B'$. So, WLOG we may pick B to be F-stable.

Stratification of X

Let B', B'' be two Borel subgroups of G. We say B' and B'' are in relative position $w \in W$ iff. there exists some $g \in G$ such that $B' = g \cdot B$ and $B'' = (g\tilde{W}) \cdot B$ (here $\tilde{w}$ is a lift of w to $N_G(T)$).

Note that if we look at $X \times X$, then from Amal's talk, we have

$$X \times X \cong \sqcup_{w \in W} \mathscr{O}(w)$$

where $\mathscr{O}(w) = G \cdot (B, \tilde{w} \cdot B)$. As a result, we have that B', B'' are in relative position w iff. $(B', B'') \in \mathscr{O}(w)$.

There is an alternative characterization of $\mathscr{O}(w)$ as the set

$$\mathscr{O}(w) = \{(g_1 \cdot B, g_2 \cdot B) : g_1^{-1} g_2 \in B\tilde{w}B\}$$

Now, $F : X \to X$ gives us its graph $\Gamma_F \subset X \times X$. We then define,

$$X(w) := \Gamma_F \cap \mathscr{O}(w)$$

One should note that $X(w)$ is not empty for any $w \in W$ by surjectivity of the Lang map. Indeed, given w, find some $g \in G$ such that $g^{-1}F(g) = w$. Then, $F(gBg^{-1}) = (gw)F(B)(gw)^{-1}$.

One can check that this intersection is transverse, so it is smooth, and since $\dim \mathscr{O}(w) = \dim X + l(w)$, we find that $\dim(X(w)) = l(w)$.

Note that if you have $(B', F(B')) \in X(w)$, then for any $g \in G^F$, $g \cdot (B', F(B')) = (g \cdot B', F(g \cdot B')) \in \Gamma_F$ and naturally $g \cdot (B', F(B')) \in \mathscr{O}(w)$. Thus, $X(w)$ admits a left action of G^F.

Note that

$$X(w) = \{gB \in X : g^{-1}F(g) \in B\tilde{w}B\} \subset X$$

and in fact,

$$X = \sqcup_{w \in W} X(w)$$

Thus, we have a stratification of X which is respected by the action of G^F. Therefore, we may work over any $X(w)$ individually.

The Deligne-Lusztig Variety

We now define $Y := G/U$ where U is the unipotent radical of B (think upper triangular matrices with 1 on the diagonal). There is then a natural map,

$$Y = G/U \to G/B = X$$

and note that since $B = T \ltimes U$, T normalizes U in B, and so Y admits a right-action by T over X. In other (slight cooler/more complicated) words, Y is a T-torsor over X (think T-bundle).

We then define

$$Y(w) = \{gU : g^{-1}F(g) \in U\tilde{w}U\} \subset Y$$

Note that $Y(w)$ lies over $X(w)$, and it admits a left G^F action which is equivariant with respect to $\pi_w : Y(w) \to X(w)$.

Also note that if $t \in T$, then $gtU \in Y(w)$ for $gU \in Y(w)$, iff. $t^{-1}g^{-1}F(g)F(t) \in U\tilde{w}U$. But then since $gU \in Y(w)$, we have $t^{-1}g^{-1}F(g)F(t) \in t^{-1}U\tilde{w}UF(t) = U(t^{-1}\tilde{w}F(t))U$. By Bruhat decomposition, we therefore find that $gtU \in Y(w)$ iff. $\mathrm{ad}(\tilde{w})(F(t)) = t$. Letting $F_w = \mathrm{ad}(\tilde{w}) \circ F$, we therefore find that $Y(w)$ is a T^{F_w}-torsor over $X(w)$. Note that because W is finite, F_w is a Frobenius morphism.

This $Y(w)$ is what we call a Deligne-Lusztig variety (note that there is some discrepancy: Deligne and Lusztig seem to consider $X(w)$ the Deligne-Lusztig variety, and then the $Y(w)$ are additional varieties over $X(w)$ which give representations [slightly unclear]).

Also note, everywhere, we've put $Y(w)$ and not $Y(\tilde{w})$ even though we use $\tilde{w}$ in the definition of $Y(w)$. This is actually okay, because if $\tilde{w}' = \tilde{w}t$ for some $t \in T$, then by finding some $t_1 \in T$ such that $t = \mathrm{ad}(\tilde{w}^{-1})(t_1)$, we can check that $gU \mapsto gt_1U$ gives an isomorphism from $Y(\tilde{w})$ to $Y(\tilde{w}')$. Therefore, we may speak of $Y(w)$.

So, since $Y(w)$ is equipped with a left-action by G^F and a right action by T^{F_w}, these two groups also act on the l-adic cohomology of $Y(w)$, and so we get a (G^F, T^{F_w})-bimodule, and so we can decompose the cohomology of $Y(w)$ via characters θ of T^{F_w}.

Thus, for every character θ of T^{F_w}, we get an induced virtual representation,

$$R_\theta = \sum_i (-1)^i H_c^i(Y(w), \overline{\mathbb{Q}_l})[\theta]$$

This is the so-called Deligne-Lusztig induced representation of the character θ (more on this next week).

This may devastate you, as twisted Frobenii might be unpleasant to handle. But we will be able to give an isomorphic construction description which is a T^F-torsor.

Rough idea:

We may express, $X(w)$ as

$$X(w) := \{g \in G : g^{-1}F(g) \in \tilde{w}U\}/T^{F_w}(U \cap \tilde{w}U\tilde{w}^{-1})$$

and similarly express,

$$Y(w) := \{g \in G : g^{-1}F(g) \in \tilde{w}U\}/(U \cap \tilde{w}U\tilde{w}^{-1})$$

We may then define,

$$X_{T,B} := \{g \in G : g^{-1}F(g) \in F(B)\}/(B \cap F(B))$$

$$Y_{T,B} := \{g \in G : g^{-1}F(g) \in F(U)\}/(U \cap F(U))$$

Then, $Y_{T,B} \to X_{T,B}$ is a T^F-torsor.

(See next week's talk for more detail, we won't need it for this week).

Recovering the Drinfeld Curve

Let us look at the case of $G = \mathrm{SL}_2$. Then, of course $G^F = \mathrm{SL}_2(\mathbb{F}_q)$, B is the group of upper triangular matrices, U the group of upper triangular matrices with the diagonal being having 1. The torus is clear. The Weyl group here is the 2-element group $< 1, w >$ where $w_{ij} = 0$ if $i = j$ and $w_{21} = -w_{12} = 1$. Using the fact that $G/U \cong \mathbb{A}^2 - (0,0)$, it is not hard to check from definitions that $Y(w)$ is the Drinfeld curve.

(Actual talk should spell out how this works).

The author of these notes is not aware of a more intrinsic way of seeing this realization (say, via flags).

Quasi-Affinity of Deligne-Lusztig Varieties

The Deligne-Lusztig varieties we have produced are, in fact, quasi-affine (i.e. there is an immersion into an affine scheme). We sketch a proof below. For more details, see *Representation Theory of Finite Reductive Groups* (link on website).

Sidebar

If we have a variety X/k ($k = \bar{k}$) equipped with a free G action, then let us assume that we may form the quotient variety X/G. We have a functor,

$$\text{Fin- } k[G]\text{-Mod} \to \text{Coh}(X/G)$$

In other words, to every finite dimensional (over k) $k[G]$-module M, we can associate a coherent $\mathscr{O}_{X/G}$-module $\mathscr{L}_{X/G}(M)$. On an affine open G-stable subset $U = \text{Spec}A$, we have that,

$$\mathscr{L}_{X/G}(M)|_U = (M \otimes_{k[G]} A)^G$$

where we give the tensor product the diagonal action.

We state a few more properties of this functor in the special case that $X \xrightarrow{\pi} X/G$ is locally trivial (i.e. on some open cover U_i of X/G, $\pi^{-1}(U_i) \cong U_i \times G$).

$$\mathscr{L}_{X/G}(\check{M}) \cong (\mathscr{L}_{X/G}(M))\check{}$$

$$\mathscr{L}_{X/G}(M \otimes N) \cong \mathscr{L}_{X/G}(M) \otimes_{\mathscr{O}_{X/G}} \mathscr{L}_{X/G}(N)$$

Pullback Theorem: Suppose $\alpha : G' \subset G$ is a subgroup, and we are given X equipped with a G-action and X' equipped with a G'-action. Suppose further that there is an α-equivariant morphism $\phi : X' \to X$ (i.e. $\phi(xg') = \phi(x)\alpha(g')$). This descends to a map,

$$\overline{\phi} : X'/G' \to X/G$$

Then, we have $\overline{\phi}^*(\mathscr{L}_{X/G}(M)) \cong \mathscr{L}_{X'/G'}(M^\alpha)$, where M^α denotes the k-vector space M equipped with the canonical G' action coming from α.

End of Sidebar

Now, returning to the proof, for every character of the torus $\lambda \in X(T)$, we can look at the one-dimensional k-vector space it generates (viewed as a T-module). We can then further view it as a one-dimensional B-module by first projecting to $T \subset B$. This gives us coherent sheaves,

$$\mathscr{L}_{G/B}(\lambda)$$

on G/B. Hereafter, we switch to additive notation for $X(T)$ (i.e. $\lambda_1 + \lambda_2 := \lambda_1\lambda_2$). Then, since $G \to G/B$ is locally trivial, we have that $\mathscr{L}_{G/B}(\lambda)$ is invertible for all λ.

Let $j : X(w) \to X = G/B$. We then claim that $j^*\mathscr{L}_{G/B}(\lambda \circ F) \cong j^*\mathscr{L}_{G/B}(\lambda(\text{ad}(w)))$.

From this claim, we will be able to prove that the structure sheaf of $X(w)$ is ample (which is equivalent to $X(w)$ being quasi-affine).

How does this follow? Well, $t \mapsto \text{ad}(w)(t)^{-1}F(t)$ is surjective from T to T by Lang's theorem, and hence the dual map $X(T) \to X(T)$,

$$\lambda \mapsto \lambda \circ \text{ad}(w) - \lambda \circ F$$

is injective. But this means the map of lattices has finite cokernel.

Now, from some magic, we can find some $\omega \in X(T)$ such that $\mathscr{L}_{G/B}(\omega)$ is ample (see *Representations of Algebraic Groups* by Jens Jantzen (II. 4.3 - 4.4) for a proof).

Thus, for some $m \in \mathbb{N}$, we can find a $\lambda \in X(T)$ such that $\lambda \circ \text{ad}(w) - \lambda \circ F = m\omega$. But $m\omega$ is still ample. Therefore, its pullback to $X(w)$ is ample, and now its straightforward to conclude, via the claim above, that $\mathscr{O}_{X(w)}$ is ample.

The proof of the claim is a fairly technical application of the pullback theorem (see *Representation Theory of Finite Reductive Groups* for more details).

We end by remarking that, to the author's knowledge, it is an open conjecture that Deligne-Lusztig varieties are actually affine. This is known for sufficiently large q.